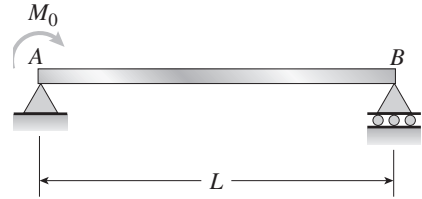


Castigliano's Theorem

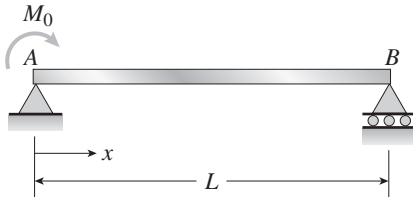
The beams described in the problems for Section 9.9 have constant flexural rigidity EI .

Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A . (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad (\text{downward})$$

$$\begin{aligned} M &= M_0 - R_A x = M_0 - \frac{M_0 x}{L} \\ &= M_0 \left(1 - \frac{x}{L}\right) \end{aligned}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI} \int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \frac{M_0^2 L}{6EI}$$

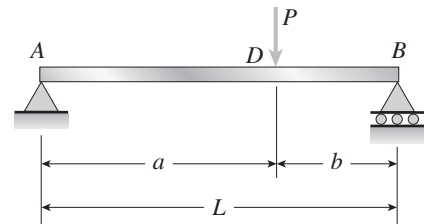
$$\text{CASTIGLIANO'S THEOREM} \quad \theta_A = \frac{dU}{dM_0} = \frac{M_0 L}{3EI} \quad (\text{clockwise})$$



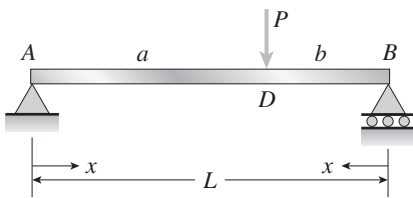
(This result agrees with Case 7, Table G-2)

Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \quad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

$$\text{STRAIN ENERGY} \quad U = \int \frac{M^2 dx}{2EI}$$

$$U_{AD} = \frac{1}{2EI} \int_0^a \left(\frac{Pbx}{L}\right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI} \int_0^b \left(\frac{Pax}{L}\right)^2 dx = \frac{P^2 a^2 b^3}{6EIL^2}$$

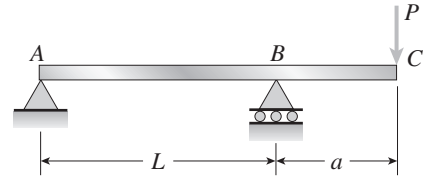
$$U = U_{AD} + U_{DB} = \frac{P^2 a^2 b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

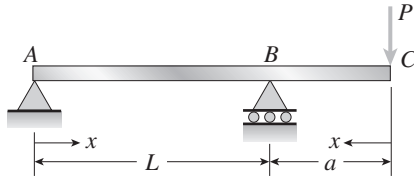
$$\delta_D = \frac{dU}{dP} = \frac{P a^2 b^2}{3LEI} \quad (\text{downward}) \quad \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a .

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 Overhanging beam



$$R_A = \frac{Pa}{L} \quad (\text{downward})$$

$$M_{AB} = -R_A x = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

STRAIN ENERGY $U = \int \frac{M^2 dx}{2EI}$

$$U_{AB} = \frac{1}{2EI} \int_0^L \left(-\frac{Pax}{L} \right)^2 dx = \frac{P^2 a^2 L}{6EI}$$

$$U_{CB} = \frac{1}{2EI} \int_0^a (-Px)^2 dx = \frac{P^2 a^3}{6EI}$$

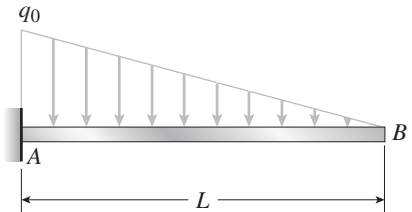
$$U = U_{AB} + U_{CB} = \frac{P^2 a^2}{6EI} (L + a)$$

CASTIGLIANO'S THEOREM

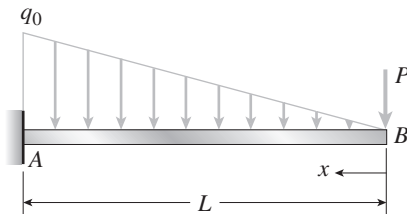
$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI} (L + a) \quad (\text{downward}) \quad \leftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0 .

Determine the deflection δ_B at the free end B . (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 Cantilever beam with triangular load



P = fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^L \left(-Px - \frac{q_0 x^3}{6L} \right)^2 dx$$

$$= \frac{P^2 L^3}{6EI} + \frac{Pq_0 L^4}{30EI} + \frac{q_0^2 L^5}{42EI}$$

CASTIGLIANO'S THEOREM

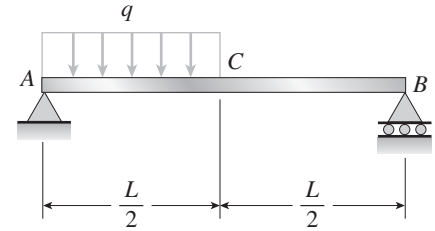
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0 L^4}{30EI} \quad (\text{downward})$$

(This result agrees with Cases 1 and 8 of Table G-1.)

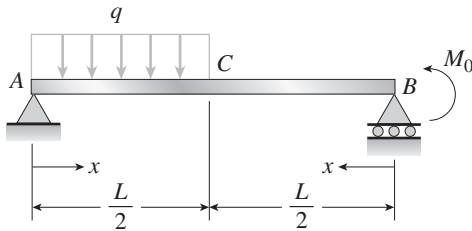
$$\text{SET } P = 0: \quad \delta_B = \frac{q_0 L^4}{30EI} \quad \leftarrow$$

Problem 9.9-5 A simple beam ACB supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B . (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-5 Simple beam with partial uniform load



M_0 = fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \quad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L} \right) x + M_0 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

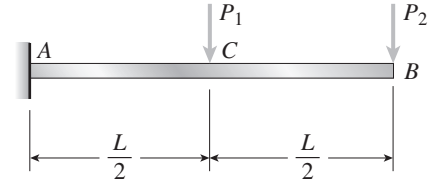
$$\begin{aligned} \theta_B &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_0} \right) dx \\ &= \frac{1}{EI} \int_0^{L/2} \left[\left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2} \right] \left[\frac{x}{L} \right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/2} \left[\left(\frac{qL}{8} - \frac{M_0}{L} \right) x + M_0 \right] \left[1 - \frac{x}{L} \right] dx \end{aligned}$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

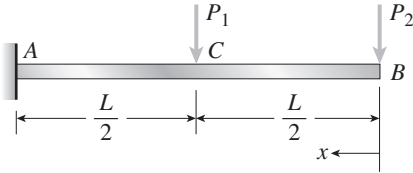
$$\begin{aligned} \theta_B &= \frac{1}{EI} \int_0^{L/2} \left(\frac{3qLx}{8} - \frac{qx^2}{2} \right) \left(\frac{x}{L} \right) dx \\ &\quad + \frac{1}{EI} \int_0^{L/2} \left(\frac{qLx}{8} \right) \left(1 - \frac{x}{L} \right) dx \\ &= \frac{qL^3}{128EI} + \frac{qL^3}{96EI} = \frac{7qL^3}{384EI} \quad (\text{counterclockwise}) \quad \leftarrow \\ &\text{(This result agrees with Case 2, Table G-2.)} \end{aligned}$$

Problem 9.9-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 , as shown in the figure.

Determine the deflections δ_C and δ_B at points C and B , respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 Cantilever beam with loads P_1 and P_2



BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT CB

$$M_{CB} = -P_2x \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \quad \frac{\partial M_{CB}}{\partial P_2} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AC

$$M_{AC} = -P_1\left(x - \frac{L}{2}\right) - P_2x \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \quad \frac{\partial M_{AC}}{\partial P_2} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\begin{aligned} \delta_C &= \frac{1}{EI} \int_0^{L/2} (M_{CB}) \left(\frac{\partial M_{CB}}{\partial P_1} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L (M_{AC}) \left(\frac{\partial M_{AC}}{\partial P_1} \right) dx \\ &= 0 + \frac{1}{EI} \int_{L/2}^L \left[-P_1\left(x - \frac{L}{2}\right) - P_2x \right] \left(\frac{L}{2} - x \right) dx \\ &= \frac{L^3}{48EI} (2P_1 + 5P_2) \quad \leftarrow \end{aligned}$$

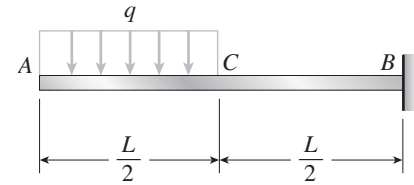
MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_B

$$\begin{aligned} \delta_B &= \frac{1}{EI} \int_0^{L/2} (M_{CB}) \left(\frac{\partial M_{CB}}{\partial P_2} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L (M_{AC}) \left(\frac{\partial M_{AC}}{\partial P_2} \right) dx \\ &= \frac{1}{EI} \int_0^{L/2} (-P_2x)(-x) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L \left[-P_1\left(x - \frac{L}{2}\right) - P_2x \right] (-x) dx \\ &= \frac{P_2L^3}{24EI} + \frac{L^3}{48EI} (5P_1 + 14P_2) \\ &= \frac{L^3}{48EI} (5P_1 + 16P_2) \quad \leftarrow \end{aligned}$$

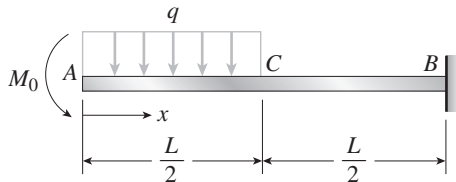
(These results can be verified with the aid of Cases 4 and 5, Table G-1.)

Problem 9.9-7 The cantilever beam ACB shown in the figure is subjected to a uniform load of intensity q acting between points A and C .

Determine the angle of rotation θ_A at the free end A . (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = -M_0 - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = -M_0 - \frac{qL}{2} \left(x - \frac{L}{4}\right) \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\begin{aligned} \theta_A &= \int \left(\frac{M}{EI}\right) \left(\frac{\partial M}{\partial M_0}\right) dx \\ &= \frac{1}{EI} \int_0^{L/2} \left(-M_0 - \frac{qx^2}{2}\right) (-1) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L \left[-M_0 - \frac{qL}{2} \left(x - \frac{L}{4}\right)\right] (-1) dx \end{aligned}$$

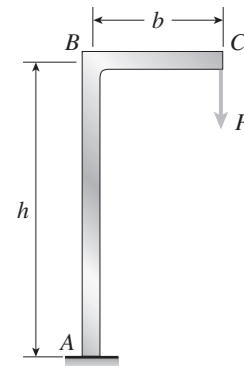
SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\begin{aligned} \theta_A &= \frac{1}{EI} \int_0^{L/2} \frac{qx^2}{2} dx + \frac{1}{EI} \int_{L/2}^L \left(\frac{qL}{2}\right) \left(x - \frac{L}{4}\right) dx \\ &= \frac{qL^3}{48EI} + \frac{qL^3}{8EI} \\ &= \frac{7qL^3}{48EI} \quad (\text{counterclockwise}) \quad \leftarrow \end{aligned}$$

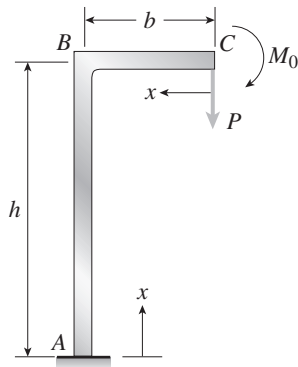
(This result can be verified with the aid of Case 3, Table G-1.)

Problem 9.9-8 The frame ABC supports a concentrated load P at point C (see figure). Members AB and BC have lengths h and b , respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end C of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C
(corresponding to the deflection δ_C)

M_0 = fictitious moment corresponding to the angle of rotation θ_C

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER AB

$$M_{AB} = Pb + M_0 \quad (0 \leq x \leq h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \quad \frac{\partial M_{AB}}{\partial M_0} = 1$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER BC

$$M_{BC} = Px + M_0 \quad (0 \leq x \leq b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \quad \frac{\partial M_{BC}}{\partial M_0} = 1$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\begin{aligned} \delta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial P} \right) dx \\ &= \frac{1}{EI} \int_0^h (Pb + M_0)(b) dx + \frac{1}{EI} \int_0^b (Px + M_0)(x) dx \end{aligned}$$

Set $M_0 = 0$:

$$\begin{aligned} \delta_C &= \frac{1}{EI} \int_0^h Pb^2 dx + \frac{1}{EI} \int_0^b Px^2 dx \\ &= \frac{Pb^2}{3EI} (3h + b) \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_C

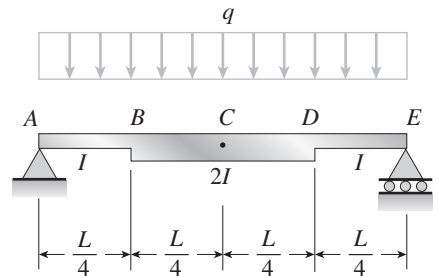
$$\begin{aligned} \theta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_0} \right) dx \\ &= \frac{1}{EI} \int_0^h (Pb + M_0)(1) dx + \frac{1}{EI} \int_0^b (Px + M_0)(1) dx \end{aligned}$$

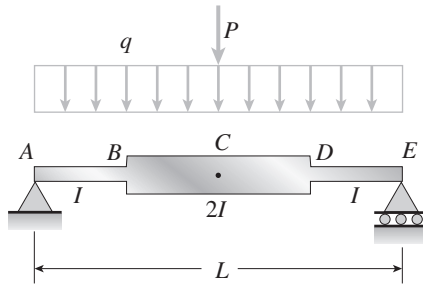
Set $M_0 = 0$:

$$\begin{aligned} \theta_C &= \frac{1}{EI} \int_0^h Pb dx + \frac{1}{EI} \int_0^b Px dx \\ &= \frac{Pb}{2EI} (2h + b) \quad (\text{clockwise}) \quad \leftarrow \end{aligned}$$

Problem 9.9-9 A simple beam $ABCDE$ supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (BCD) is twice the moment of inertia in the end parts (AB and DE).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 Nonprismatic beam

P = fictitious load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR THE LEFT-HAND HALF OF THE BEAM (A TO C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

Integrate from A to C and multiply by 2.

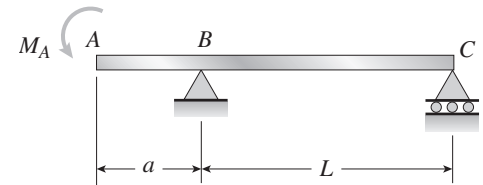
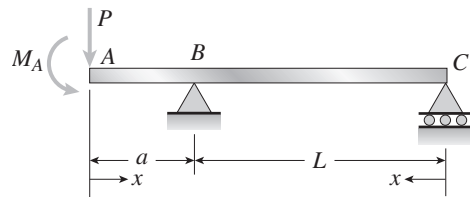
$$\begin{aligned} \delta_C &= 2 \int \left(\frac{M_{AC}}{EI} \right) \left(\frac{\partial M_{AC}}{\partial P} \right) dx \\ &= 2 \left(\frac{1}{EI} \right) \int_0^{L/4} \left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \right) \left(\frac{x}{2} \right) dx \\ &\quad + 2 \left(\frac{1}{2EI} \right) \int_{L/4}^{L/2} \left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \right) \left(\frac{x}{2} \right) dx \end{aligned}$$

SET FICTITIOUS LOAD P EQUAL TO ZERO

$$\begin{aligned} \delta_C &= \frac{2}{EI} \int_0^{L/4} \left(\frac{qLx}{2} - \frac{qx^2}{2} \right) \left(\frac{x}{2} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/4}^{L/2} \left(\frac{qLx}{2} - \frac{qx^2}{2} \right) \left(\frac{x}{2} \right) dx \\ &= \frac{13 qL^4}{6,144 EI} + \frac{67 qL^4}{12,288 EI} \\ \delta_C &= \frac{31 qL^4}{4096 EI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

Problem 9.9-10 An overhanging beam ABC is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L , respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A. (Obtain the solution by using the modified form of Castigliano's theorem.)

**Solution 9.9-10 Overhanging beam ABC**

M_A = couple acting at the free end A (corresponding to the angle of rotation θ_A)

P = fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \leq x \leq a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \quad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT BC

$$\text{Reaction at support C: } R_C = \frac{M_A}{L} + \frac{Pa}{L} \quad (\text{downward})$$

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \leq x \leq L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \quad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_A

$$\begin{aligned}\theta_A &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_A} \right) dx \\ &= \frac{1}{EI} \int_0^a (-M_A - Px)(-1) dx \\ &\quad + \frac{1}{EI} \int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L} \right) \left(-\frac{x}{L} \right) dx\end{aligned}$$

Set $P = 0$:

$$\begin{aligned}\theta_A &= \frac{1}{EI} \int_0^a M_A dx + \frac{1}{EI} \int_0^L \left(\frac{M_A x}{L} \right) \left(\frac{x}{L} \right) dx \\ &= \frac{M_A}{3EI} (L + 3a) \quad (\text{counterclockwise}) \quad \leftarrow\end{aligned}$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_A

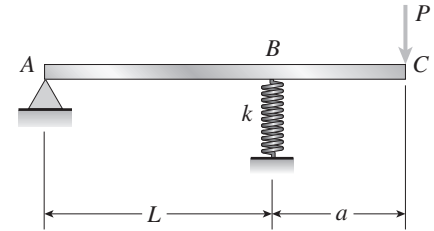
$$\begin{aligned}\delta_A &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial P} \right) dx \\ &= \frac{1}{EI} \int_0^a (-M_A - Px)(-x) dx \\ &\quad + \frac{1}{EI} \int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L} \right) \left(-\frac{ax}{L} \right) dx\end{aligned}$$

Set $P = 0$:

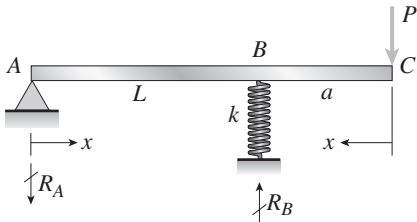
$$\begin{aligned}\delta_A &= \frac{1}{EI} \int_0^a M_A x dx + \frac{1}{EI} \int_0^L \left(\frac{M_A x}{L} \right) \left(\frac{ax}{L} \right) dx \\ &= \frac{M_A a}{6EI} (2L + 3a) \quad (\text{downward}) \quad \leftarrow\end{aligned}$$

Problem 9.9-11 An overhanging beam ABC rests on a simple support at A and a spring support at B (see figure). A concentrated load P acts at the end of the overhang. Span AB has length L , the overhang has length a , and the spring has stiffness k .

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 Beam with spring support



$$R_A = \frac{Pa}{L} \quad (\text{downward})$$

$$R_B = \frac{P}{L} (L + a) \quad (\text{upward})$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AB

$$M_{AB} = -R_A x = -\frac{Pax}{L} \quad \frac{dM_{AB}}{dP} = -\frac{ax}{L} \quad (0 \leq x \leq L)$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT BC

$$M_{BC} = -Px \quad \frac{dM_{BC}}{dP} = -x \quad (0 \leq x \leq a)$$

STRAIN ENERGY OF THE SPRING (EQ. 2-38a)

$$U_s = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2kL^2}$$

STRAIN ENERGY OF THE BEAM (EQ. 9-80a)

$$U_B = \int \frac{M^2 dx}{2EI}$$

TOTAL STRAIN ENERGY U

$$U = U_B + U_s = \int \frac{M^2 dx}{2EI} + \frac{P^2(L + a)^2}{2kL^2}$$

APPLY CASTIGLIANO'S THEOREM (EQ. 9-87)

$$\begin{aligned}\delta_C &= \frac{dU}{dP} = \frac{d}{dP} \int \frac{M^2 dx}{2EI} + \frac{d}{dP} \left[\frac{P^2(L + a)^2}{2kL^2} \right] \\ &= \frac{d}{dP} \int \frac{M^2 dx}{2EI} + \frac{P(L + a)^2}{kL^2}\end{aligned}$$

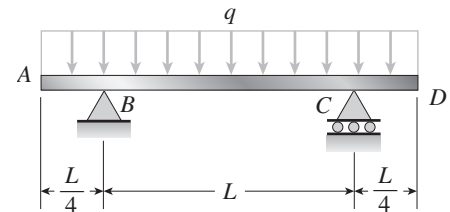
DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\begin{aligned}\delta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{dM}{dP} \right) dx + \frac{P(L+a)^2}{kL^2} \\ &= \frac{1}{EI} \int_0^L \left(-\frac{Pax}{L} \right) \left(-\frac{ax}{L} \right) dx\end{aligned}$$

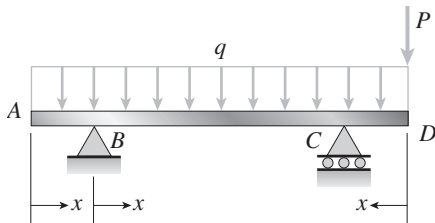
$$\begin{aligned}&+ \frac{1}{EI} \int_0^a (-Px)(-x) dx + \frac{P(L+a)^2}{kL^2} \\ &= \frac{Pa^2L}{3EI} + \frac{Pa^3}{3EI} + \frac{P(L+a)^2}{kL^2} \\ \delta_C &= \frac{Pa^2(L+a)}{3EI} + \frac{P(L+a)^2}{kL^2} \quad \leftarrow\end{aligned}$$

Problem 9.9-12 A symmetric beam $ABCD$ with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 Beam with overhangs



q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3qL}{4} - \frac{P}{4} \quad R_C = \frac{3qL}{4} + \frac{5P}{4}$$

BENDING MOMENTS AND PARTIAL DERIVATIVES

SEGMENT AB

$$M_{AB} = -\frac{qx^2}{2} \quad \frac{\partial M_{AB}}{\partial P} = 0 \quad \left(0 \leq x \leq \frac{L}{4} \right)$$

SEGMENT BC

$$\begin{aligned}M_{BC} &= -\left[q \left(x + \frac{L}{4} \right) \right] \left[\frac{1}{2} \left(x + \frac{L}{4} \right) \right] + R_B x \\ &= -\frac{q}{2} \left(x + \frac{L}{4} \right)^2 + \left(\frac{3qL}{4} - \frac{P}{4} \right) x \quad (0 \leq x \leq L)\end{aligned}$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

$$\text{SEGMENT } CD \quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \leq x \leq \frac{L}{4} \right)$$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_D

$$\begin{aligned}\delta_D &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial P} \right) dx \\ &= \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2} \right) (0) dx \\ &\quad + \frac{1}{EI} \int_0^L \left[-\frac{q}{2} \left(x + \frac{L}{4} \right)^2 + \left(\frac{3qL}{4} - \frac{P}{4} \right) x \right] \left[-\frac{x}{4} \right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2} - Px \right) (-x) dx\end{aligned}$$

SET $P = 0$:

$$\begin{aligned}\delta_D &= \frac{1}{EI} \int_0^L \left[-\frac{q}{2} \left(x + \frac{L}{4} \right)^2 + \frac{3qL}{4} x \right] \left[-\frac{x}{4} \right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2} \right) (-x) dx \\ &= -\frac{5qL^4}{768EI} + \frac{qL^4}{2048EI} = -\frac{37qL^4}{6144EI}\end{aligned}$$

(Minus means the deflection is opposite in direction to the fictitious load P .)

$$\therefore \delta_D = \frac{37qL^4}{6144EI} \quad (\text{upward}) \quad \leftarrow$$

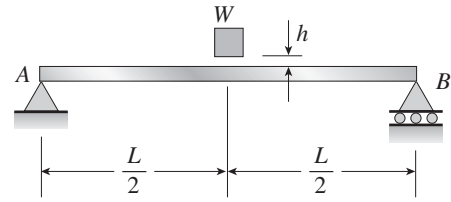
Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI . Disregard the weights of the beams themselves, and consider only the effects of the given loads.

Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress σ_{\max} due to the falling weight in terms of h , σ_{st} , and δ_{st} , where σ_{st} is the maximum bending stress and δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio $\sigma_{\max}/\sigma_{\text{st}}$ (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st} . (Let h/δ_{st} vary from 0 to 10.)



Solution 9.10-1 Weight W dropping onto a simple beam

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{\max} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

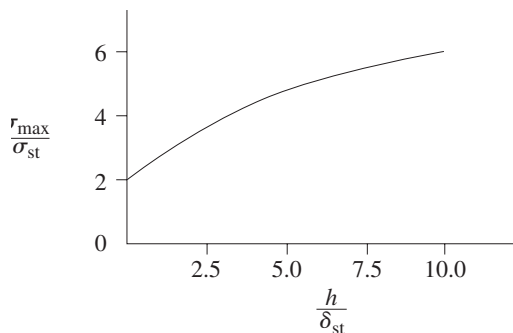
$$\therefore \frac{\sigma_{\max}}{\sigma_{\text{st}}} = \frac{\delta_{\max}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2}$$

$$\sigma_{\max} = \sigma_{\text{st}} \left[1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \right] \quad \leftarrow$$

$\frac{h}{\delta_{\text{st}}}$	$\frac{\sigma_{\max}}{\sigma_{\text{st}}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

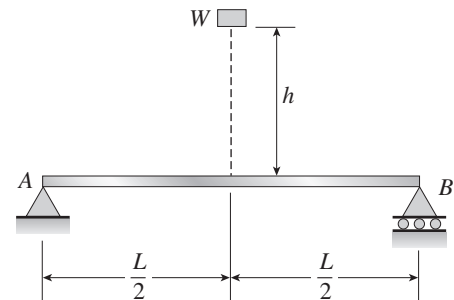
NOTE: $\delta_{\text{st}} = \frac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

GRAPH OF RATIO $\sigma_{\max}/\sigma_{\text{st}}$



Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A .

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress σ_{\max} in the beam due to the falling weight.



Solution 9.10-2 Weight W dropping onto a simple beam

Height h is very large.

MAXIMUM DEFLECTION (EQ. 9-95)

$$\delta_{\max} = \sqrt{2h\delta_{\text{st}}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\begin{aligned} \therefore \frac{\sigma_{\max}}{\sigma_{\text{st}}} &= \frac{\delta_{\max}}{\delta_{\text{st}}} = \sqrt{\frac{2h}{\delta_{\text{st}}}} \\ \sigma_{\max} &= \sqrt{\frac{2h\sigma_{\text{st}}^2}{\delta_{\text{st}}}} \end{aligned} \quad (1)$$

$$\begin{aligned} \sigma_{\text{st}} &= \frac{M}{S} = \frac{WL}{4S} & \sigma_{\text{st}}^2 &= \frac{W^2L^2}{16S^2} \\ \delta_{\text{st}} &= \frac{WL^3}{48EI} & \frac{\sigma_{\text{st}}^2}{\delta_{\text{st}}} &= \frac{3WEI}{S^2L} \end{aligned} \quad (2)$$

For a RECTANGULAR BEAM (with b , depth d):

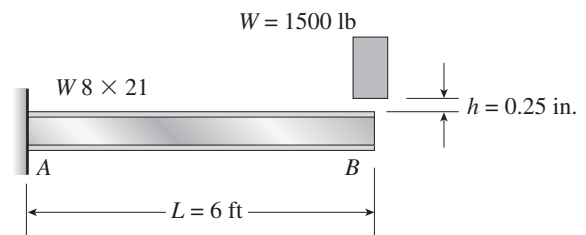
$$I = \frac{bd^3}{12} \quad S = \frac{bd^2}{6} \quad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \quad (3)$$

Substitute (2) and (3) into (1):

$$\sigma_{\max} = \sqrt{\frac{18WhE}{AL}} \quad \leftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a $W 8 \times 21$ wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection δ_{\max} of the end of the beam and the maximum bending stress σ_{\max} due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)

**Solution 9.10-3 Cantilever beam**

DATA: $L = 6$ ft = 72 in. $W = 1500$ lb
 $h = 0.25$ in. $E = 30 \times 10^6$ psi
 $W 8 \times 21$ $I = 75.3$ in.⁴ $S = 18.2$ in.³

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{\text{st}} = W/k$, where k is the stiffness of the particular structure being considered. For instance: Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \frac{3EI}{L^3}$ Etc.

For the cantilever beam in this problem:

$$\begin{aligned} \delta_{\text{st}} &= \frac{WL^3}{3EI} = \frac{(1500 \text{ lb})(72 \text{ in.})^3}{3(30 \times 10^6 \text{ psi})(75.3 \text{ in.}^4)} \\ &= 0.08261 \text{ in.} \end{aligned}$$

Equation (9-94):

$$\delta_{\max} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2} = 0.302 \text{ in.} \quad \leftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

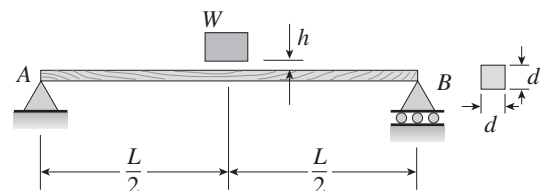
$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{PL}{S} \quad \delta_{\max} = \frac{PL^3}{3EI}$$

$$\text{Ratio: } \frac{\sigma_{\max}}{\delta_{\max}} = \frac{3EI}{SL^2}$$

$$\therefore \sigma_{\max} = \frac{3EI}{SL^2} \delta_{\max} = 21,700 \text{ psi} \quad \leftarrow$$

Problem 9.10-4 A weight $W = 20$ kN falls through a height $h = 1.0$ mm onto the midpoint of a simple beam of length $L = 3$ m (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12$ GPa.

If the allowable bending stress in the wood is $\sigma_{\text{allow}} = 10$ MPa, what is the minimum required dimension d ?



Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20 \text{ kN}$ $h = 1.0 \text{ mm}$ $L = 3.0 \text{ m}$
 $E = 12 \text{ GPa}$ $\sigma_{\text{allow}} = 10 \text{ MPa}$

CROSS SECTION OF BEAM (SQUARE)

$d =$ dimension of each side

$$I = \frac{d^4}{12} \quad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{\text{st}} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \quad (2)$$

$$\delta_{\text{st}} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{\text{max}}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10 \text{ MPa})d^3}{3(20 \text{ kN})(3.0 \text{ m})} = 1 + \left[1 + \frac{8(1.0 \text{ mm})(12 \text{ GPa})d^4}{(20 \text{ kN})(3.0 \text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

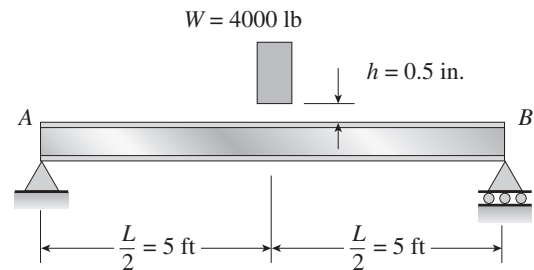
$$d = 0.2804 \text{ m} = 280.4 \text{ mm}$$

For minimum value, round upward.

$$\therefore d = 281 \text{ mm} \quad \leftarrow$$

Problem 9.10-5 A weight $W = 4000 \text{ lb}$ falls through a height $h = 0.5 \text{ in.}$ onto the midpoint of a simple beam of length $L = 10 \text{ ft}$ (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{\text{allow}} = 18,000 \text{ psi}$ and $E = 30 \times 10^6 \text{ psi}$, select the lightest wide-flange beam listed in Table E-1 in Appendix E that will be satisfactory.

**Solution 9.10-5 Simple beam of wide-flange shape**

DATA: $W = 4000 \text{ lb}$ $h = 0.5 \text{ in.}$
 $L = 10 \text{ ft} = 120 \text{ in.}$
 $\sigma_{\text{allow}} = 18,000 \text{ psi}$ $E = 30 \times 10^6 \text{ psi}$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

$$\text{or } \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{\text{st}} = \frac{M}{S} = \frac{WL}{4S} \quad \delta_{\text{st}} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \sigma_{\text{allow}} \left(\frac{4S}{WL}\right) = \frac{4\sigma_{\text{allow}}S}{WL} \quad (2)$$

$$\frac{2h}{\delta_{\text{st}}} = 2h \left(\frac{48EI}{WL^3}\right) = \frac{96hEI}{WL^3} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{\text{allow}}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{\text{allow}}} \left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}\right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3} \text{ in.}^3 \right) \left[1 + \left(1 + \frac{5I}{24} \right)^{1/2} \right] \quad (4)$$

($S = \text{in.}^3$; $I = \text{in.}^4$)

PROCEDURE

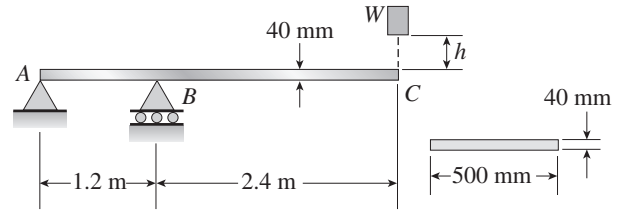
1. Select a trial beam from Table E-1.
2. Substitute I into Eq. (4) and calculate required S .
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual		Required S
	I	S	
W 8 × 35	127	31.2	41.6 (NG)
W 10 × 45	248	49.1	55.0 (NG)
W 10 × 60	341	66.7	63.3 (OK)
W 12 × 50	394	64.7	67.4 (NG)
W 14 × 53	541	77.8	77.8 (OK)
W 16 × 31	375	47.2	66.0 (NG)

Lightest beam is W 14 × 53 ←

Problem 9.10-6 An overhanging beam ABC of rectangular cross section has the dimensions shown in the figure. A weight $W = 750 \text{ N}$ drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa , what is the maximum height h from which the weight may be dropped? (Assume $E = 12 \text{ GPa}$.)

**Solution 9.10-6 Overhanging beam**

DATA: $W = 750 \text{ N}$ $L_{AB} = 1.2 \text{ m}$ $L_{BC} = 2.4 \text{ m}$
 $E = 12 \text{ GPa}$ $\sigma_{\text{allow}} = 45 \text{ MPa}$

$$I = \frac{bd^3}{12} = \frac{1}{12} (500 \text{ mm})(40 \text{ mm})^3$$

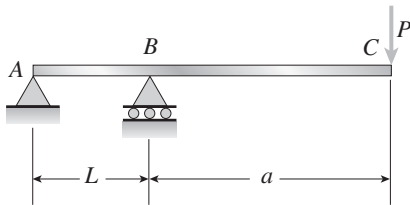
$$= 2.6667 \times 10^6 \text{ mm}^4$$

$$= 2.6667 \times 10^{-6} \text{ m}^4$$

$$S = \frac{bd^2}{6} = \frac{1}{6} (500 \text{ mm})(40 \text{ mm})^2$$

$$= 133.33 \times 10^3 \text{ mm}^3$$

$$= 133.33 \times 10^{-6} \text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG $P =$ load at end C $L =$ length of span AB $a =$ length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L+a)}{3EI}$$

$$\text{Stiffness of the beam: } k = \frac{P}{\delta_C} = \frac{3EI}{a^2(L+a)} \quad (1)$$

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{\text{st}} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

$$\text{Simple beam with load at midpoint: } k = \frac{48EI}{L^3}$$

$$\text{Cantilever beam with load at free end: } k = \frac{3EI}{L^3} \text{ Etc.}$$

For the overhanging beam in this problem (see Eq. 1):

$$\delta_{\text{st}} = \frac{W}{k} = \frac{Wa^2(L+a)}{3EI} \quad (2)$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{\text{st}} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3EI} \quad (3)$$

EQUATION (9-94):

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

or

$$\frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}} \right)^{1/2} \quad (4)$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\max}}{\sigma_{\text{st}}} = \frac{\delta_{\max}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (5)$$

$$\sigma_{\text{st}} = \frac{M}{S} = \frac{WL_{BC}}{S} \quad (6)$$

MAXIMUM HEIGHT h

Solve Eq. (5) for h :

$$\frac{\sigma_{\max}}{\sigma_{\text{st}}} - 1 = \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2}$$

$$\left(\frac{\sigma_{\max}}{\sigma_{\text{st}}}\right)^2 - 2\left(\frac{\sigma_{\max}}{\sigma_{\text{st}}}\right) + 1 = 1 + \frac{2h}{\delta_{\text{st}}}$$

$$h = \frac{\delta_{\text{st}}}{2} \left(\frac{\sigma_{\max}}{\sigma_{\text{st}}}\right) \left(\frac{\sigma_{\max}}{\sigma_{\text{st}}} - 2\right) \quad (7)$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for σ_{\max} :

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} \left(\frac{\sigma_{\text{allow}}S}{WL_{BC}}\right) \left(\frac{\sigma_{\text{allow}}S}{WL_{BC}} - 2\right) \quad (8)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} = 0.08100 \text{ m}$$

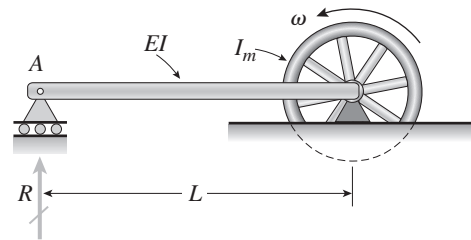
$$\frac{\sigma_{\text{allow}}S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100 \text{ m}) \left(\frac{10}{3}\right) \left(\frac{10}{3} - 2\right) = 0.36 \text{ m}$$

$$\text{or } h = 360 \text{ mm} \quad \leftarrow$$

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?

**Solution 9.10-7 Rotating flywheel**

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$kE = \frac{1}{2} I_m \omega^2$$

$$\text{STRAIN ENERGY OF BEAM } U = \int \frac{M^2 dx}{2EI}$$

$M = Rx$, where x is measured from support A .

$$U = \frac{1}{2EI} \int_0^L (Rx)^2 dx = \frac{R^2 L^3}{6EI}$$

CONSERVATION OF ENERGY

$$kE = U \quad \frac{1}{2} I_m \omega^2 = \frac{R^2 L^3}{6EI}$$

$$R = \sqrt{\frac{3EI I_m \omega^2}{L^3}} \quad \leftarrow$$

NOTE: The moment of inertia I_M has units of $\text{kg} \cdot \text{m}^2$ or $\text{N} \cdot \text{m} \cdot \text{s}^2$